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Superfield integrals in high dimensions

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Abstract: We present an efficient, covariant, graph-based method to integrate superfields over fermionic spaces of high dimensionality. We illustrate this method with the computation of the most general sixteen-dimensional Majorana-Weyl integral in ten dimensions. Our method has applications to the construction of higher-derivative supergravity actions as well as the computation of string and membrane vertex operator correlators.

Keywords: Superspaces, Extended Supersymmetry, Supersymmetric Effective Theories.

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Contents

1. Introduction

Despite the conceptual elegance of superspace methods, their use in the study of supergravity and string theory is hampered by a number of technical difficulties. The problem which is perhaps most manifest is the fact that high-dimensional fermionic integrals, although conceptually simple, are hard to evaluate explicitly. This forms a serious obstacle when one attempts to relate the often elegant superspace expressions to results in terms of supergravity component fields.

The fermionic integration problem is encountered in a variety of different situations. One of these is the construction of ten-dimensional higher-derivative supergravity actions. By virtue of the presence of the dilaton field, these supergravity theories allow for the existence of scalar superfields which contain the entire supergravity multiplet in their component expansion [1, 2]. This observation has led (now almost twenty years ago) to the hope that complicated higher-derivative actions can perhaps be constructed in terms of superspace integrals of simple expressions [3, 4]. Although the application of this idea to the type-IIB theory is beset with difficulties [5], even the simpler construction in $N = 1$ supergravity has never been worked out in full detail. One particular problem which has remained unsolved is how to relate superspace expressions to component ones. This same problem also appears in the computation of vertex operator correlators for superparticles [6], strings and membranes [7], when these are formulated using target-space spinors (i.e. using the Green-Schwarz or Berkovits formalisms). High-dimensional fermionic integrals appear here when one tries to integrate out fermionic zero modes of the fields living on the world-volume.

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While generic covariant fermionic integrals are thus so far not known, special cases which exhibit additional symmetries are sometimes tractable. One such case is the integral that leads to the sixteen-dilatino interaction in the type-IIB theory [8, 9]. The superspace supergravity computation and the computation of a sixteen-fermion vertex operator correlator both lead to the trivial integral

$$\int \mathrm{d}^{16}\theta \left(\theta_a \lambda^a\right)^{16} = \epsilon_{a_1 \cdots a_{16}} \lambda^{a_1} \cdots \lambda^{a_{16}} = 16!\, \lambda^1 \cdots \lambda^{16}\,. \tag{1.1}$$

A slightly more complicated expression is obtained for the four-graviton amplitude in the Green-Schwarz formalism, when computed in the light-cone gauge. The resulting fermionic integral can be decomposed as the product of two known SO(8) fermionic integrals [10], with the result [11]

$$\int \mathrm{d}^8\vartheta \mathrm{d}^8\dot{\vartheta} \Big((\bar{\vartheta}\gamma^{mn}\vartheta)(\bar{\dot{\vartheta}}\gamma^{rs}\dot{\vartheta}) R_{mn\,rs} \Big)^4 = \left(t_8 t_8 - \frac{1}{4}\epsilon_8\epsilon_8 \right) R^4\,. \tag{1.2}$$

Again, this integral also occurs in an analysis of four-graviton couplings in supergravity. A similar result can also be obtained for the four-point correlator of supermembrane vertex operators in the light-cone gauge [7]. However, it is clear that (1.1) and (1.2) form only the tip of the iceberg. Many interesting results wait to be derived once a fully covariant way is established to perform an arbitrary high-dimensional fermionic superintegral and express it in terms of Lorentz singlets (Kronecker deltas and epsilon tensors).

Therefore, it is the purpose of this small note to discuss a generic, covariant method for the integration of arbitrary functions over high-dimensional fermionic spaces. We demonstrate the feasibility of our method by deriving an explicit expression for the most general sixteen-component SO(10) fermionic integral (the "$N = 1$ integral") in terms of Lorentz singlets. This result is rather interesting by itself, and we will discuss our motivation to derive it, including possible applications, in some more detail towards the end (in section 3). The method can be applied easily to the other ten- and eleven-dimensional supergravity theories. Expressions for SO(9) integrals, relevant for superparticle and supermembrane calculations in eleven-dimensional supergravity in the light-cone gauge, will appear shortly [12].

For completeness, we also describe, in the appendix, an efficient method to reduce tensor polynomials to a minimal basis. This simple method does not seem to be widely known, but is of considerable help in dealing with higher-derivative Lorentz invariants.

2. Fermionic integrals

2.1 A simple eight-fermion example

The goal of this note is to show how high-dimensional fermionic integrals can be evaluated covariantly and in full generality. However, the techniques which we will use apply also to much simpler cases. It is therefore illustrative to first consider a simpler fermionic integral, which can be done by hand and for which the answer has been known for a long time, so as to get familiar with the techniques.

Let us thus consider the following integral over the eight-dimensional space of SO(8) spinors [10],

$$I_{\pm}^{i_1 j_1 \cdots i_4 j_4} := \int \mathrm{d}^8\theta^{\pm} \left(\theta^{\pm}\gamma^{i_1 j_1}\theta^{\pm}\right) \cdots \left(\theta^{\pm}\gamma^{i_4 j_4}\theta^{\pm}\right). \tag{2.1}$$

Here the $\pm$ symbols denote the chirality of the spinors. In order to determine the number of Lorentz singlets which is needed to express this integral, we compute the tensor product of the four symmetrised sets of two anti-symmetric vector indices [13],

$$\mathrm{Sym}^4\left([010\ldots]\right) = (\boxminus)^4_{\text{sym}} = \begin{cases} 3 \times [0000] \oplus \cdots & \text{in SO(8)}, \\ 2 \times [00000\ldots] \oplus \cdots & \text{in SO}(2k) \text{ for } k > 4. \end{cases} \tag{2.2}$$

This result implies that (2.1) can be decomposed in two delta singlets and one epsilon singlet (the epsilon singlet is dimension dependent and corresponds to the disappearing singlet when the tensor product is evaluated in higher dimensions). It is straightforward to find these three independent singlets; we will use

$$\begin{aligned} D_1 &= \frac{1}{12}\left(\delta^{i_1 i_2}\delta^{j_1 j_2}\delta^{i_3 i_4}\delta^{j_3 j_4} + 11 \text{ terms}\right), \\ D_2 &= \frac{1}{48}\left(\delta^{j_1 i_2}\delta^{j_2 i_3}\delta^{j_3 i_4}\delta^{j_4 i_1} + 47 \text{ terms}\right), \\ E &= \varepsilon^{i_1 j_1 i_2 \cdots j_4}. \end{aligned} \tag{2.3}$$

The fermionic integral can thus be written as

$$I_{\pm}^{i_1 j_1 \cdots i_4 j_4} = \alpha_1\, D_1^{i_1 j_1 \cdots i_4 j_4} + \alpha_2\, D_2^{i_1 j_1 \cdots i_4 j_4} \pm \beta\, E^{i_1 j_1 i_2 \cdots j_4}, \tag{2.4}$$

and the goal is to determine the unknown coefficients α_1, α_2 and β.

By using an explicit representation for the SO(8) gamma matrices, it is straightforward to evaluate (2.1) for particular values of the eight indices (by selecting the terms in the resulting polynomial of spinor components in which each component occurs once). Similarly, it is straightforward to determine the value of D_1, D_2 and E for a particular set of index values. Three independent combinations are listed below.

$[i_1 j_1]\cdots\ [i_4 j_4]$	I_+	I_-	D_1	D_2	E
[12][12][34][34]	−128	−128	1/12	0	0
[12][23][34][41]	128	128	0	1/48	0
[12][34][56][78]	128	−128	0	0	1

(2.5)

This leads to three equations for three unknowns, from which one determines the coefficients to be

$$\alpha_1 = -1536, \qquad \alpha_2 = 6144, \qquad \beta = 128. \tag{2.6}$$

Comparing with the t_8 tensor of [10, appendix 9.A], one then obtains the expected result

$$I_{\pm} = 256\, t_{8,\pm}. \tag{2.7}$$

In the next section we will see that the sixteen fermion integral can be evaluated using precisely the same logic, although the number of Lorentz singlets increases sharply and it also becomes more complicated to evaluate their values given a set of indices. This increased complexity calls for a number of new ideas.

2.2 The sixteen fermion integral

Let us now turn to the evaluation of the sixteen fermion integral in ten-dimensional simple supergravity. The Weyl and Majorana properties of the spinor θ imply that bilinears in θ can be written as three-forms. The most general integrand therefore has the form

$$I^{i_1 j_1 k_1 \cdots i_8 j_8 k_8} := \int \mathrm{d}^{16}\theta \, (\bar\theta \Gamma^{i_1 j_1 k_1}\theta)(\bar\theta \Gamma^{i_2 j_2 k_2}\theta) \cdots (\bar\theta \Gamma^{i_8 j_8 k_8}\theta)\,. \tag{2.8}$$

The goal will again be to express this integral in terms of Lorentz singlets, i.e. Kronecker deltas and epsilon tensors carrying the free vector indices. That is, we want to write the integral as

$$I^{i_1 j_1 k_1 \cdots i_8 j_8 k_8} = \sum_i \alpha_i \, T_{(i)}^{i_1 j_1 k_1 \cdots i_8 j_8 k_8}\,, \tag{2.9}$$

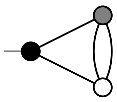
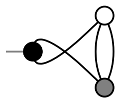
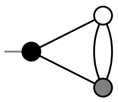
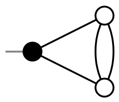
in terms of a set of basis tensors $T_{(i)}$.

The first step in this integration is to determine the number of Lorentz singlets in which the integral (2.8) can be decomposed. This number is easily obtained by considering the tensor product

$$\mathrm{Sym}^8\left([00100\ldots]\right) = \left(\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array}\right)^8_{\mathrm{sym}} = \begin{cases} 33 \times [00000] \oplus \cdots & \text{in } \mathrm{SO}(10), \\ 24 \times [000\ldots] \oplus \cdots & \text{in } \mathrm{SO}(2k) \text{ for } k > 5. \end{cases} \tag{2.10}$$

There are thus 33 singlets in SO(10). Nine of these disappear when one considers higher-dimensional spaces, and these thus correspond to parity-odd invariants involving the ten-dimensional epsilon tensor.

In contrast to the situation in the previous section, it is now not so easy to guess the 33 Lorentz singlets. However, the explicit construction of these singlets can be translated to an elegant problem in graph theory. Let us first focus on the singlets which involve only Kronecker deltas. We will represent fully anti-symmetrised index triplets by trivalent nodes, and represent Kronecker deltas which set two indices equal by edges connecting the nodes. Multiple edges are allowed (e.g. $\delta^{i_1 i_2}\delta^{j_1 j_2}\delta^{k_1 k_2}$, which corresponds to two nodes with a 3-fold connection between them). The problem of finding all 24 independent singlets now corresponds, in standard graph terminology, to the problem of finding all 3-regular multigraphs (not necessarily connected) with eight vertices. All graphs of this type can be found with the help of [14, 15], and one finds a total of 32 graphs. This may seem to contradict (2.10), however, the anti-symmetry of the index triplets is not yet fully encoded at this stage. Using anti-symmetry, 8 of the 32 graphs can be shown to correspond to a vanishing expression. For example, the double contraction of two identical three-forms is symmetric in the two free indices and therefore vanishes when contracted into a further three-form. In graphical notation, this is seen from

 $=$  $= -$  $\Rightarrow$  $= 0\,.$ (2.11)

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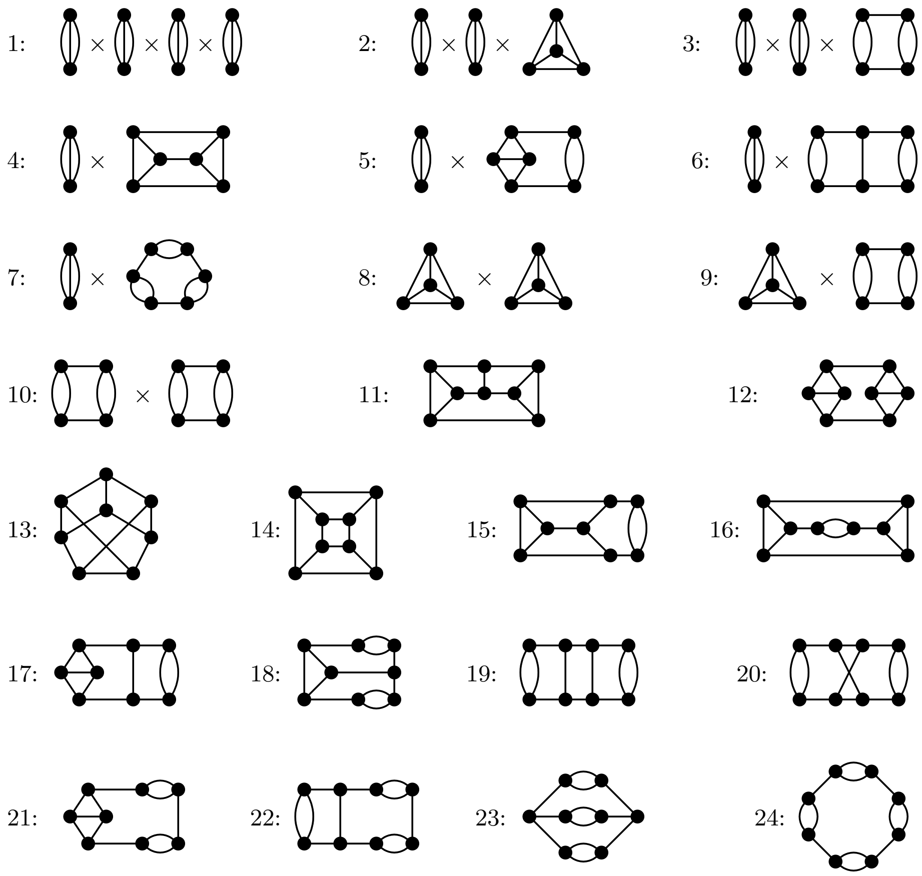
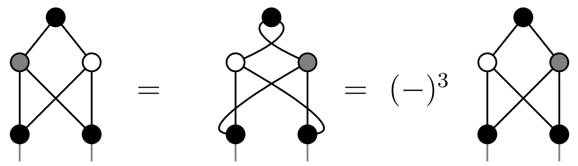


Figure 1: Graphical display of the 24 Lorentz singlets in the θ^{16} integral which can be expressed solely using Kronecker deltas. Each dot represents a fully anti-symmetrised index triplet (e.g. $[i_1, j_1, k_1]$) and the lines indicate how these indices appear on Kronecker delta symbols.

Another vanishing subgraph is



(2.12)

These identities make 8 graphs vanish identically, see figure 2. The remaining 24 graphs are listed in figure 1 and their explicit expressions in terms of Kronecker deltas can be found in table 1.

The construction of the graphs corresponding to parity-odd singlets is less systematic. We can again introduce a graphical notation, representing an epsilon tensor by a 10-valent vertex (or "box"). Many graphs can be formed out of one 10-valent and eight trivalent vertices, but nine of these are sufficient to represent the nine independent parity-odd singlets in the tensor product (2.10). These are displayed in figure 3 and the corresponding explicit expressions can be found in table 2.

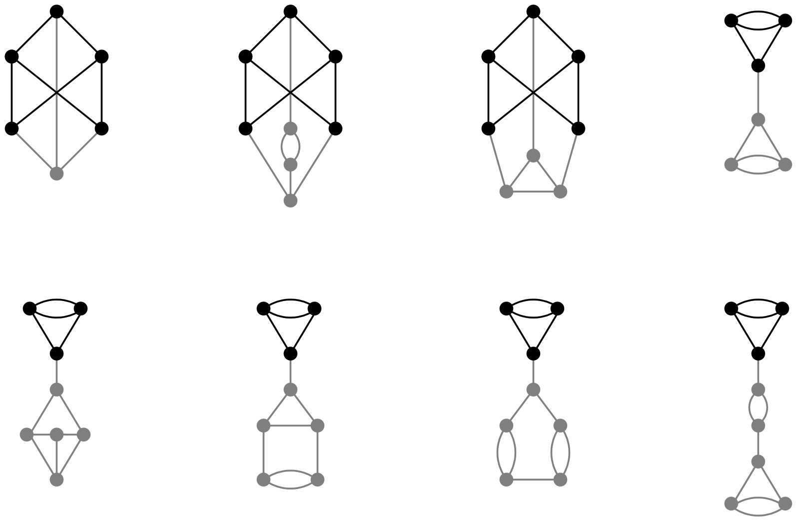


Figure 2: Graphs of parity-even type which vanish identically by virtue of the identities (2.11) and (2.12).

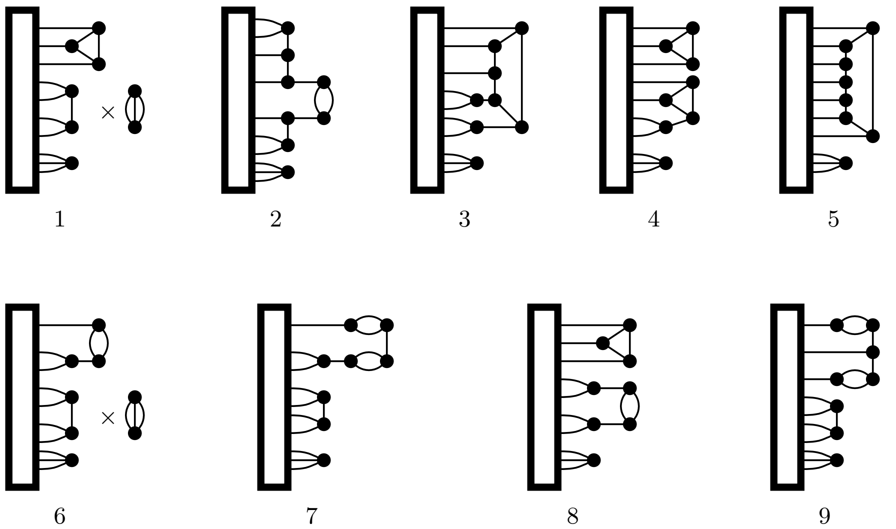


Figure 3: Graphical display of the 9 Lorentz singlets in the θ^{16} integral which contain epsilon tensors. Rectangles denote epsilon tensors, and each dot represents a fully anti-symmetrised index triplet, as in figure 1.

Having determined the basis on which (2.8) can be decomposed, the remaining step is to determine the coefficients in front of each of these basis tensors. This is again done by matching the values of (2.8) and (2.9) for various sets of values of the indices, and solving the resulting system of linear equations.[1] In graph-theory language, the evaluation of (2.9) corresponds to finding all ways of colouring the edges of a graph with numbers,

[1] The chirality of the spinor θ in (2.8) enters at this stage. As in section 2.1, the choice of chirality reflects itself in the sign of the coefficients $\alpha_{25} \ldots \alpha_{33}$ of the parity-odd singlets, and we will not comment on this any further.

Graph i	Singlet $T_{(i)}$	Coefficient $\alpha_i/(2^{19}\cdot 3^6)$
1	$\delta^{i_1}_{i_2}\,\delta^{i_3}_{i_4}\,\delta^{i_5}_{i_6}\,\delta^{i_7}_{i_8}\,\delta^{j_1}_{j_2}\,\delta^{j_3}_{j_4}\,\delta^{j_5}_{j_6}\,\delta^{j_7}_{j_8}\,\delta^{k_1}_{k_2}\,\delta^{k_3}_{k_4}\,\delta^{k_5}_{k_6}\,\delta^{k_7}_{k_8}$	-269
2	$\delta^{i_1}_{i_2}\,\delta^{i_3}_{i_4}\,\delta^{i_5}_{i_6}\,\delta^{i_7}_{j_6}\,\delta^{i_8}_{k_5}\,\delta^{j_1}_{j_2}\,\delta^{j_3}_{j_4}\,\delta^{j_5}_{j_7}\,\delta^{j_8}_{k_6}\,\delta^{k_1}_{k_2}\,\delta^{k_3}_{k_4}\,\delta^{k_7}_{k_8}$	4968
3	$\delta^{i_1}_{i_2}\,\delta^{i_3}_{i_4}\,\delta^{i_5}_{i_6}\,\delta^{i_7}_{k_6}\,\delta^{i_8}_{k_5}\,\delta^{j_1}_{j_2}\,\delta^{j_3}_{j_4}\,\delta^{j_5}_{j_6}\,\delta^{j_7}_{j_8}\,\delta^{k_1}_{k_2}\,\delta^{k_3}_{k_4}\,\delta^{k_7}_{k_8}$	7956
4	$\delta^{i_1}_{i_2}\,\delta^{i_3}_{i_4}\,\delta^{i_5}_{j_4}\,\delta^{i_6}_{k_3}\,\delta^{i_7}_{j_3}\,\delta^{i_8}_{k_5}\,\delta^{j_1}_{j_2}\,\delta^{j_5}_{j_6}\,\delta^{j_7}_{k_4}\,\delta^{j_8}_{k_6}\,\delta^{k_1}_{k_2}\,\delta^{k_7}_{k_8}$	-2304
5	$\delta^{i_1}_{i_2}\,\delta^{i_3}_{i_4}\,\delta^{i_5}_{j_3}\,\delta^{i_6}_{k_4}\,\delta^{i_7}_{k_3}\,\delta^{i_8}_{k_6}\,\delta^{j_1}_{j_2}\,\delta^{j_4}_{j_5}\,\delta^{j_6}_{k_5}\,\delta^{j_7}_{j_8}\,\delta^{k_1}_{k_2}\,\delta^{k_7}_{k_8}$	70848
6	$\delta^{i_1}_{i_2}\,\delta^{i_3}_{i_4}\,\delta^{i_5}_{k_3}\,\delta^{i_6}_{k_4}\,\delta^{i_7}_{k_5}\,\delta^{i_8}_{k_6}\,\delta^{j_1}_{j_2}\,\delta^{j_3}_{j_4}\,\delta^{j_5}_{j_6}\,\delta^{j_7}_{j_8}\,\delta^{k_1}_{k_2}\,\delta^{k_7}_{k_8}$	-24192
7	$\delta^{i_1}_{i_2}\,\delta^{i_3}_{i_4}\,\delta^{i_5}_{k_4}\,\delta^{i_6}_{j_5}\,\delta^{i_7}_{k_6}\,\delta^{i_8}_{j_7}\,\delta^{j_1}_{j_2}\,\delta^{j_3}_{j_4}\,\delta^{j_6}_{k_5}\,\delta^{j_8}_{k_7}\,\delta^{k_1}_{k_2}\,\delta^{k_3}_{k_8}$	-32544
8	$\delta^{i_1}_{i_2}\,\delta^{i_3}_{j_2}\,\delta^{i_4}_{k_1}\,\delta^{i_5}_{i_6}\,\delta^{i_7}_{j_6}\,\delta^{i_8}_{k_5}\,\delta^{j_1}_{j_3}\,\delta^{j_4}_{k_2}\,\delta^{j_5}_{j_7}\,\delta^{j_8}_{k_6}\,\delta^{k_3}_{k_4}\,\delta^{k_7}_{k_8}$	-3888
9	$\delta^{i_1}_{i_2}\,\delta^{i_3}_{j_2}\,\delta^{i_4}_{k_1}\,\delta^{i_5}_{i_6}\,\delta^{i_7}_{k_6}\,\delta^{i_8}_{k_5}\,\delta^{j_1}_{j_3}\,\delta^{j_4}_{k_2}\,\delta^{j_5}_{j_6}\,\delta^{j_7}_{j_8}\,\delta^{k_3}_{k_4}\,\delta^{k_7}_{k_8}$	-26352
10	$\delta^{i_1}_{i_2}\,\delta^{i_3}_{k_2}\,\delta^{i_4}_{k_1}\,\delta^{i_5}_{i_6}\,\delta^{i_7}_{k_6}\,\delta^{i_8}_{k_5}\,\delta^{j_1}_{j_2}\,\delta^{j_3}_{j_4}\,\delta^{j_5}_{j_6}\,\delta^{j_7}_{j_8}\,\delta^{k_3}_{k_4}\,\delta^{k_7}_{k_8}$	-20412
11	$\delta^{i_1}_{i_2}\,\delta^{i_3}_{j_1}\,\delta^{i_4}_{k_3}\,\delta^{i_5}_{k_1}\,\delta^{i_6}_{k_4}\,\delta^{i_7}_{k_2}\,\delta^{i_8}_{k_5}\,\delta^{j_2}_{j_3}\,\delta^{j_4}_{j_5}\,\delta^{j_6}_{j_8}\,\delta^{j_7}_{k_6}\,\delta^{k_7}_{k_8}$	124416
12	$\delta^{i_1}_{i_2}\,\delta^{i_3}_{j_1}\,\delta^{i_4}_{k_2}\,\delta^{i_5}_{k_1}\,\delta^{i_6}_{j_5}\,\delta^{i_7}_{k_5}\,\delta^{i_8}_{k_4}\,\delta^{j_2}_{j_3}\,\delta^{j_4}_{k_3}\,\delta^{j_6}_{j_7}\,\delta^{j_8}_{k_6}\,\delta^{k_7}_{k_8}$	10368
13	$\delta^{i_1}_{i_2}\,\delta^{i_3}_{j_1}\,\delta^{i_4}_{k_1}\,\delta^{i_5}_{j_2}\,\delta^{i_6}_{k_4}\,\delta^{i_7}_{k_3}\,\delta^{i_8}_{k_2}\,\delta^{j_3}_{j_6}\,\delta^{j_4}_{j_5}\,\delta^{j_7}_{k_5}\,\delta^{j_8}_{k_6}\,\delta^{k_7}_{k_8}$	196992
14	$\delta^{i_1}_{i_2}\,\delta^{i_3}_{j_2}\,\delta^{i_4}_{j_3}\,\delta^{i_5}_{k_1}\,\delta^{i_6}_{k_2}\,\delta^{i_7}_{k_3}\,\delta^{i_8}_{k_4}\,\delta^{j_1}_{j_4}\,\delta^{j_5}_{j_6}\,\delta^{j_7}_{k_6}\,\delta^{j_8}_{k_7}\,\delta^{k_5}_{k_8}$	-10368
15	$\delta^{i_1}_{i_2}\,\delta^{i_3}_{j_1}\,\delta^{i_4}_{k_3}\,\delta^{i_5}_{k_2}\,\delta^{i_6}_{k_1}\,\delta^{i_7}_{k_6}\,\delta^{i_8}_{k_5}\,\delta^{j_2}_{j_3}\,\delta^{j_4}_{j_5}\,\delta^{j_6}_{k_4}\,\delta^{j_7}_{j_8}\,\delta^{k_7}_{k_8}$	373248
16	$\delta^{i_1}_{i_2}\,\delta^{i_3}_{j_1}\,\delta^{i_4}_{k_3}\,\delta^{i_5}_{j_4}\,\delta^{i_6}_{k_5}\,\delta^{i_7}_{k_2}\,\delta^{i_8}_{k_1}\,\delta^{j_2}_{j_3}\,\delta^{j_5}_{k_4}\,\delta^{j_6}_{j_7}\,\delta^{j_8}_{k_6}\,\delta^{k_7}_{k_8}$	-331776
17	$\delta^{i_1}_{i_2}\,\delta^{i_3}_{j_1}\,\delta^{i_4}_{k_2}\,\delta^{i_5}_{k_4}\,\delta^{i_6}_{k_1}\,\delta^{i_7}_{k_5}\,\delta^{i_8}_{k_6}\,\delta^{j_2}_{j_3}\,\delta^{j_4}_{k_3}\,\delta^{j_5}_{j_6}\,\delta^{j_7}_{j_8}\,\delta^{k_7}_{k_8}$	-165888
18	$\delta^{i_1}_{i_2}\,\delta^{i_3}_{j_1}\,\delta^{i_4}_{j_2}\,\delta^{i_5}_{j_3}\,\delta^{i_6}_{k_1}\,\delta^{i_7}_{k_4}\,\delta^{i_8}_{k_5}\,\delta^{j_4}_{k_2}\,\delta^{j_5}_{k_3}\,\delta^{j_6}_{j_7}\,\delta^{j_8}_{k_6}\,\delta^{k_7}_{k_8}$	41472
19	$\delta^{i_1}_{i_2}\,\delta^{i_3}_{k_1}\,\delta^{i_4}_{k_2}\,\delta^{i_5}_{k_3}\,\delta^{i_6}_{k_4}\,\delta^{i_7}_{k_5}\,\delta^{i_8}_{k_6}\,\delta^{j_1}_{j_2}\,\delta^{j_3}_{j_4}\,\delta^{j_5}_{j_6}\,\delta^{j_7}_{j_8}\,\delta^{k_7}_{k_8}$	-10368
20	$\delta^{i_1}_{i_2}\,\delta^{i_3}_{k_1}\,\delta^{i_4}_{k_2}\,\delta^{i_5}_{k_3}\,\delta^{i_6}_{j_3}\,\delta^{i_7}_{k_5}\,\delta^{i_8}_{k_6}\,\delta^{j_1}_{j_2}\,\delta^{j_4}_{j_6}\,\delta^{j_5}_{k_4}\,\delta^{j_7}_{j_8}\,\delta^{k_7}_{k_8}$	-171072
21	$\delta^{i_1}_{i_2}\,\delta^{i_3}_{j_1}\,\delta^{i_4}_{k_2}\,\delta^{i_5}_{k_1}\,\delta^{i_6}_{j_5}\,\delta^{i_7}_{k_6}\,\delta^{i_8}_{k_4}\,\delta^{j_2}_{j_3}\,\delta^{j_4}_{k_3}\,\delta^{j_6}_{k_5}\,\delta^{j_7}_{j_8}\,\delta^{k_7}_{k_8}$	-238464
22	$\delta^{i_1}_{i_2}\,\delta^{i_3}_{k_1}\,\delta^{i_4}_{k_2}\,\delta^{i_5}_{k_3}\,\delta^{i_6}_{k_4}\,\delta^{i_7}_{j_5}\,\delta^{i_8}_{k_7}\,\delta^{j_1}_{j_2}\,\delta^{j_3}_{j_4}\,\delta^{j_6}_{j_8}\,\delta^{j_7}_{k_5}\,\delta^{k_6}_{k_8}$	248832
23	$\delta^{i_1}_{i_2}\,\delta^{i_3}_{j_1}\,\delta^{i_4}_{k_1}\,\delta^{i_5}_{i_8}\,\delta^{i_6}_{j_8}\,\delta^{i_7}_{k_8}\,\delta^{j_2}_{j_5}\,\delta^{j_3}_{j_6}\,\delta^{j_4}_{j_7}\,\delta^{k_2}_{k_5}\,\delta^{k_3}_{k_6}\,\delta^{k_4}_{k_7}$	-62208
24	$\delta^{i_1}_{i_2}\,\delta^{i_3}_{k_2}\,\delta^{i_4}_{j_3}\,\delta^{i_5}_{k_4}\,\delta^{i_6}_{j_5}\,\delta^{i_7}_{k_6}\,\delta^{i_8}_{j_7}\,\delta^{j_1}_{j_2}\,\delta^{j_4}_{k_3}\,\delta^{j_6}_{k_5}\,\delta^{j_8}_{k_7}\,\delta^{k_1}_{k_8}$	63504

Table 1: The coefficients of the 24 Lorentz singlets in the θ^{16} integral which can be expressed solely using Kronecker deltas. The singlets are understood to be antisymmetrised in the $[ijk]$ triplets, and symmetrised over $1\dots8$.

given a set of three numbers at each vertex. This requires some care in order to keep the computation within bounds. The basis tensors in table 1 and 2 contain an implicit anti-symmetrisation over all indices in each triplet, as well as an implicit symmetrisation over all index triplets. A brute-force algorithm which investigates the value of a given singlet for all terms in the symmetrisation therefore leads to a worst-case situation in which $(3!)^8\cdot 8!\sim 6.8\times 10^{10}$ different terms (or colourings) have to be considered. The

Graph i	Singlet $T_{(24+i)}$	Coefficient $\alpha_{24+i}/(2^{21}\cdot 3^6\cdot 5)$
1	$\eta_{i_1i_2}\,\eta_{i_3i_4}\,\eta_{i_5j_3}\,\eta_{i_6i_7}\,\eta_{j_1j_2}\,\eta_{j_4j_5}\,\eta_{k_1k_2}\,\epsilon_{i_8j_8k_8j_7k_7j_6k_6k_5k_4k_3}$	7
2	$\eta_{i_1i_2}\,\eta_{i_3k_1}\,\eta_{i_4k_2}\,\eta_{i_5j_3}\,\eta_{i_6j_4}\,\eta_{i_7j_5}\,\eta_{j_1j_2}\,\epsilon_{i_8j_8k_8j_7k_7j_6k_6k_5k_4k_3}$	42
3	$\eta_{i_1i_2}\,\eta_{i_3j_1}\,\eta_{i_4j_2}\,\eta_{i_5j_3}\,\eta_{i_6k_1}\,\eta_{i_7k_2}\,\eta_{j_4j_5}\,\epsilon_{i_8j_8k_8j_7k_7j_6k_6k_5k_4k_3}$	−294
4	$\eta_{i_1i_2}\,\eta_{i_3j_1}\,\eta_{i_4i_5}\,\eta_{i_6j_4}\,\eta_{i_7k_1}\,\eta_{j_2j_3}\,\eta_{j_5j_6}\,\epsilon_{i_8j_8k_8j_7k_7k_6k_5k_4k_3k_2}$	−168
5	$\eta_{i_1i_2}\,\eta_{i_3j_1}\,\eta_{i_4j_2}\,\eta_{i_5j_3}\,\eta_{i_6j_4}\,\eta_{i_7j_5}\,\eta_{j_6j_7}\,\epsilon_{i_8j_8k_8k_7k_6k_5k_4k_3k_2k_1}$	264
6	$\eta_{i_1i_2}\,\eta_{i_3i_4}\,\eta_{i_5k_3}\,\eta_{i_6i_7}\,\eta_{j_1j_2}\,\eta_{j_3j_4}\,\eta_{k_1k_2}\,\epsilon_{i_8j_8k_8j_7k_7j_6k_6j_5k_5k_4}$	0
7	$\eta_{i_1i_2}\,\eta_{i_3k_1}\,\eta_{i_4j_3}\,\eta_{i_5k_2}\,\eta_{i_6i_7}\,\eta_{j_1j_2}\,\eta_{j_4k_3}\,\epsilon_{i_8j_8k_8j_7k_7j_6k_6j_5k_5k_4}$	0
8	$\eta_{i_1i_2}\,\eta_{i_3i_4}\,\eta_{i_5j_3}\,\eta_{i_6k_1}\,\eta_{i_7k_2}\,\eta_{j_1j_2}\,\eta_{j_4j_5}\,\epsilon_{i_8j_8k_8j_7k_7j_6k_6k_5k_4k_3}$	0
9	$\eta_{i_1i_3}\,\eta_{i_2j_4}\,\eta_{i_4k_1}\,\eta_{i_5j_2}\,\eta_{i_6i_7}\,\eta_{j_1j_3}\,\eta_{j_5k_2}\,\epsilon_{i_8j_8k_8j_7k_7j_6k_6k_5k_4k_3}$	0

Table 2: Coefficients of the nine selected epsilon singlets which occur in the fermionic integral (2.8). The vanishing of four of the coefficients has been achieved by choosing a suitable basis of the nine parity-odd singlets in (2.10).

graphical representation suggests a much more efficient "backtracking" algorithm. This algorithm constructs the graph labellings vertex by vertex and checks after each choice if the index assignment is still consistent. If it is, the next vertex is labelled, if not, the algorithm backtracks and proceeds to the next choice of edge labelling.

The result of this matching and the subsequent solution of the linear system is presented in the last column of tables 1 and 2. This concludes the computation of (2.8).

2.3 Covariant computation of the R^4 integral

A useful check of our method is to compute the well-known R^4 term in the linearised heterotic or type-IIB theory, and verify that it reproduces results which were obtained previously using non-covariant methods. The R^4 term arises from a superfield integral of the form

$$I_{R^4} = \int \mathrm{d}^{16}\theta\,\Phi^4\,, \tag{2.13}$$

where the relevant terms in the scalar superfield Φ take the form

$$\Phi = \cdots + (\bar\theta\Gamma^{ijm}\theta)(\bar\theta\Gamma^{kl}{}_m\theta)R_{ijkl} + \cdots\,. \tag{2.14}$$

By making explicit use of the results of the previous subsection, we can express the integral in terms of the 24 independent parity-even Lorentz singlets,

$$I_{R^4} = \sum_{i=1}^{24}\alpha_i\,T_{(i)}^{i_1j_1k_1\cdots i_8j_8k_8}\,[\eta_{i_1i_2}R_{j_1k_1j_2k_2}]\cdots[\eta_{i_7i_8}R_{j_7k_7j_8k_8}]\,. \tag{2.15}$$

(Note that the parity-odd part of the integral does not contribute, since there are no ϵR^4 scalars in ten dimensions.) The individual terms in the sum lead to lengthy linear combinations of the 26 independent quartic curvature scalars. All dependence on the Ricci curvature cancels in the total result, and by decomposing the result on the basis of the 7 Fulling invariants [16] (see also the appendix) we are left with the required result in terms of Weyl tensors,

$$\begin{aligned} I_{R^4} &= 2^{24}3^4 \left(t_8 t_8 + \frac{1}{8}\epsilon_{10}\epsilon_{10} \right) C^4 \\ &= 2^{32}3^5 \left(-\frac{1}{4} C^{pqrs} C_{pq}{}^{tu} C_{rt}{}^{vw} C_{suvw} + C^{pqrs} C_p{}^t{}_r{}^u C_t{}^v{}_q{}^w C_{uvsw} \right). \end{aligned} \tag{2.16}$$

Imposing this result actually turns out to be restrictive enough to completely determine the coefficients $\alpha_1 \ldots \alpha_{24}$.

3. Discussion and applications

We have presented the computation of the ten-dimensional fermionic integral (2.8) in terms of 33 basis tensors, using a general method applicable to all fermionic integrals, in particular those of ten- and eleven-dimensional supergravity. The solution of this problem was facilitated by mapping it onto a graph construction and colouring problem. To conclude, let us discuss a number of applications of this integration procedure.

Let us start with the $N = 1$ heterotic theory. The use of the dilaton superfield for the construction of the R^4 invariant was proposed a long time ago [3]. Although the expansion of the scalar superfield was only worked out to lowest order, it should be possible to extend this analysis to higher order in θ using computer assistance (a similar analysis would have to be performed to construct the measure). One particular reason why it is interesting to pursue this program is that it could provide insight into the still elusive supersymmetrisation of the $(t_8t_8 + \frac{1}{8}\epsilon_{10}\epsilon_{10})R^4$ invariant. As was emphasised in [17], the t_8t_8 part of this construction is reasonably well understood because of its relation to a super-Yang-Mills invariant. The $\epsilon\epsilon R^4$ term [18], in contrast, does not admit a derivation by "squaring" a super-Yang-Mills action. However, this term potentially plays an important role in the modification of the superspace torsion constraints. A direct derivation in components would certainly help to understand this issue.

The complete set of higher derivative interactions that accompany the R^4 term involve, in particular, fluxes that are essential for understanding nontrivial compactifications of string/M-theory. Unfortunately, there are intrinsic difficulties in the superspace description of such interactions due to the absence of an off-shell superspace formalism for theories with maximal supersymmetry. However, a limited amount can be deduced from the available on-shell superspace formulations. For example, in the type-IIB theory the superfield formulation of [2] encapsulates the classical theory which contains component fields comprising the one-form (made from a derivative of the complex scalar), the NS-NS and R-R three-forms and the five-form field strength (as well as the fermions). There are difficulties in extending this to the general leading higher derivative interactions [5] but in the special case in which only the five-form and the metric are non-vanishing, the five-form dependence enters purely as a companion to the curvature in the θ^4 term in the scalar superfield [5, 19]. As shown in [19] this leads to an elegant understanding of the nonrenormalisation of the

$D3$-brane supergravity solution by these leading higher derivative interactions. In other examples string perturbation theory has provided evidence for the structure of such terms, but a superfield analysis has not yet been possible. For example, certain H^2R^3 terms were determined from a string calculation in [20, 21]. In any case, the technical results of this paper may be of value in any future progress towards an understanding of the superspace formulation of such interactions.

A completely different application of our superintegration techniques concerns the study of vertex operator correlators. In space-time supersymmetric formalisms for superparticles, strings and membranes, the leading higher-derivative amplitudes arise as simple expressions over world-sheet fields, involving only a fermionic zero-mode integral. Such integrals are precisely of the type considered here. Using the methods of the present paper, it has recently become possible to determine the $(DF_{(4)})^2R^2$ and $(DF_{(4)})^4$ terms in the M-theory effective action directly from a superparticle calculation [12].

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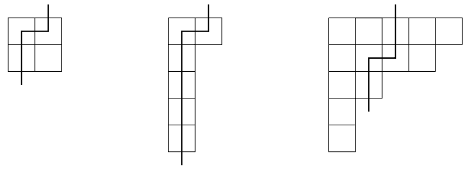
A. Reduction of tensor polynomials

Once one has done a superspace integral over a sufficiently complex integrand, one typically ends up with tensor polynomials which can be further reduced using the symmetries of the individual tensors, as well as the exchange of identical tensors. We would here like to comment briefly on a reduction method which incorporates *all* symmetries, including the Ricci cyclic identity and Bianchi identities (this method was used implicitly in several papers, see e.g. [22, 23], but as far as we know was never spelled out, and more importantly, does not seem to be widely known; it is not mentioned in the standard reference [16] and is also missed in most recent literature on tensor polynomial simplification).

The simplest symmetries of tensors are *mono-term* symmetries, such as anti-symmetry in a set of indices. These symmetries always relate one particular index distribution to one other distribution, and can be used step-by-step to reduce a given tensor monomial to a canonical form. Various efficient algorithms have been discussed in the literature and we will not comment on these symmetries any further (see [24 – 26] and in particular the implementation in [27]).

The more complicated symmetries are *multi-term* symmetries, such as the Ricci cyclic identity or the Bianchi identity, which relate a sum of terms with different index distributions. These symmetries are all manifestations of the so-called *Garnir* symmetries of Young tableaux [28]. These state that the sum over all anti-symmetrisation of boxes in a

Garnir hook is identically zero. Examples of such Garnir hooks are given below,



which represent the Ricci cyclic identity, the Bianchi identity on a five-form and a more general Garnir symmetry, respectively. Applying a Garnir symmetry on a tensor produces a sum of tensors, which means that one can no longer restrict to the canonicalisation of tensor monomials.

There is, however, a simple way to construct instead a *basis* of monomials which takes the Garnir symmetries into account, and reduce any given expression to this basis. It consists of simply replacing each tensor in a monomial by its Young-projected form. A basis of monomials can now be constructed by first generating a list of all possible index contractions of the tensors, and then projecting each of these using the Young projection on each of the individual tensors. Monomials which are identical by virtue of Garnir symmetries will then map to the same sum of monomials.

There are, however, a few subtleties. Firstly, it would be prohibitively expensive to write down all terms in a Young-projected tensor. Instead, it is much more efficient to reduce the Young-projected forms by making use of the mono-term symmetries, which are easy to deal with using the methods of [24–26]. One thus obtains, for e.g. the Riemann tensor,

$$R_{abcd} \to \frac{1}{3}\big(2\,R_{abcd} - R_{adbc} + R_{acbd}\big)\,, \tag{A.1}$$

instead of the $(2!)^4 = 16$ terms which are produced by the Young projector. The expression on the right-hand side manifestly satisfies the cyclic Ricci identity, even if one only knows about the mono-term symmetries of the Riemann tensor. Using the projector (A.1) it is easy to show e.g. that $2\,R_{abcd}R_{acbd} = R_{abcd}R_{abcd}$. The monomial on the left-hand side maps to

$$R_{abcd}R_{acbd} \to \frac{1}{3}\big(R_{abcd}R_{acbd} + R_{abcd}R_{abcd}\big)\,, \tag{A.2}$$

and similarly $R_{abcd}R_{abcd}$ maps to twice this expression, thereby proving the identity in a way which easily extends to much more complicated cases.

Secondly, the Young projectors are not dimension dependent. As a result, not all relations between monomials will be recognised. The additional relations exist because invariants sometimes arise as the contraction of tensors with two d-dimensional epsilon tensors. When written out in terms of Kronecker deltas, this leads to anti-symmetrisation in d indices when none of the indices of the epsilon tensors are contracted with each other. Clearly, such an invariant can be written down in lower dimensions, but it vanishes identically, although this is not recognised by simply performing the Young projections. Relations obtained in this way have to be taken into account separately, but are easy to find.

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